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The intersection numbers of KTSs with a common parallel class[☆]

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ABSTRACT

In this paper, the intersection numbers of KTSs with a common parallel class are investigated. Denote by $J_1[u]$ the set of all integers k such that there is a pair of KTS$(3u)$s with a common parallel class intersecting in $k+u$ triples, u of them being the triples of the common parallel class. It has been established that $J_1[u] = \{0, 1, \ldots, 3\binom{u}{2}-6, 3\binom{u}{2}-4, 3\binom{u}{2}\}$ for any odd integer $u \geq 7$ and $u \neq 9, 11, 13, 17, 19$. For $u = 9, 11, 13, 17, 19$, there are 11 cases that are left undecided.

1. Introduction

A *Steiner triple system of order* v (briefly STS(v)) is a pair $(X, \mathcal{B})$ where X is a v-set and $\mathcal{B}$ is a collection of 3-subsets of X (called triples) such that every pair of distinct elements of X belongs to exactly one triple of $\mathcal{B}$.

A *Kirkman triple system of order* v (briefly KTS(v)) is a Steiner triple system of order v $(X, \mathcal{B})$ together with a partition R of the set of triples $\mathcal{B}$ into subsets $R_1, R_2, \ldots, R_n$ called parallel classes such that each R_i $(i = 1, 2, \ldots, n)$ is a partition of X. It is well known that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$ [7], and hence a KTS$(3u)$ exists if and only if $u \equiv 1 \pmod 2$. Chang and Lo Faro [2] determined the pairs (k, v) for which there exists a pair of Kirkman triple systems on the same v-set having k triples in common with only 10 pairs of (k, v) undecided. Chang and Lo Faro [3] determined the pairs $(k, 2r+1)$ for which there exists a pair of Kirkman triple systems on the same $(2r+1)$-set having $k+r$ triples in common and r of them possessing a common element of X with only 9 pairs of $(k, 2r+1)$ undecided.

Two Kirkman triple systems of order v $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ with a common parallel class are said to intersect in k other triples provided $|\mathcal{B}_1 \cap \mathcal{B}_2| = k + v/3$. In this paper, we investigate the intersection numbers of KTSs with a common parallel class. Denote by $J_1[u]$ the set of all integers k such that there exists a pair of KTS$(3u)$s with a common parallel class having $k+u$ triples in common, u of them being the triples of the common parallel class. Let $I_1[u] = \{0, 1, \ldots, 3\binom{u}{2}-6, 3\binom{u}{2}-4, 3\binom{u}{2}\}$. By [5], it is obvious to see that $J_1[u] \subseteq I_1[u]$. In this paper, we will determine the set $J_1[u]$ for any odd integer $u \geq 3$ and $u \neq 9, 11, 13, 17, 19$. For $u = 9, 11, 13, 17, 19$, there are 11 cases that are left undecided.

2. Auxiliary designs and recursive constructions

In this section, we shall introduce some terminology and describe some auxiliary designs to be used in our constructions.

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Let K be a set of positive integers. A K-GDD with design $(X, \mathcal{G}, \mathcal{B})$ on a v-set X consists of a family $\mathcal{G}$ of distinct subsets of X (called *groups*) which partition X, and with a collection $\mathcal{B}$ of subsets of X (called *blocks*), with $|B| \in K$ for all $B \in \mathcal{B}$, such that (i) each block contains at most one element from each group; (ii) each pair of elements from different groups occurs in exactly one block.

The *type* of the K-GDD $(X, \mathcal{G}, \mathcal{B})$ is defined to be the multiset $T = \{|G| : G \in \mathcal{G}\}$. We also use $a^i b^j c^k \cdots$ to denote the type, which means that in the multiset there are i occurrences of a, j occurrences of b, etc. A *pairwise balanced design* (PBD) $B(K, 1; v)$ is a K-GDD with group type 1^v. A $B(\{k\}, 1; v)$ is essentially a $(v, k, 1)$-BIBD. We usually write $\{k\}$-GDD as k-GDD for brevity. Furthermore, the k-GDD $(X, \mathcal{G}, \mathcal{B})$ is called *resolvable*, denoted by k-RGDD, if there exists a partition $\Gamma = \{P_1, P_2, \ldots, P_r\}$ of $\mathcal{B}$ such that each part P_i forms a *parallel class*, i.e., a partition of X. It is easy to see that a 3-RGDD with group type 3^u yields a KTS$(3u)$ if we consider the groups to be further blocks.

A *holey parallel class* of a GDD $(X, \mathcal{G}, \mathcal{B})$ is a set of blocks P which forms a partition of $X \setminus G$, for some $G \in \mathcal{G}$. The group G is called *the hole corresponding* to P. A K-frame is a K-GDD $(X, \mathcal{G}, \mathcal{B})$ in which the block set can be partitioned into holey parallel classes. We refer to a 3-frame as a Kirkman frame. Two k-frames on the same set and with the same groups $(X, \mathcal{G}, \mathcal{A})$ and $(X, \mathcal{G}, \mathcal{B})$ are said to intersect in l blocks provided $|\mathcal{A} \cap \mathcal{B}| = l$. Apply the Fundamental Frame Construction [10] to obtain the following construction. The proof can be found in [2].

Theorem 2.1 (*Fundamental Construction*)**.** *Suppose that $(X, \mathcal{G}, \mathcal{A})$ is a GDD, and w is a function from X to $Z^+ \cup \{0\}$. For every block $A \in \mathcal{A}$, suppose that there is a pair of k-frames of type $\{w(x) : x \in A\}$ with b_A blocks in common. Then, there exists a pair of k-frames of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$ having $\sum_{A \in \mathcal{A}} b_A$ blocks in common.*

Theorem 2.2. *Suppose that there are two Kirkman frames of type $\{t_1, t_2, \ldots, t_n\}$ with b common triples. For $1 \le i \le n$, suppose that $3|t_i$ and there exists a pair of 3-RGDDs of type $3^{1+t_i/3}$ containing b_i common triples. Then there exists a pair of 3-RGDDs of type $3^{1+\sum_{i=1}^{n} t_i/3}$ having $b + \sum_{i=1}^{n} b_i$ common triples.*

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ and $(X, \mathcal{G}, \mathcal{B})$ be two Kirkman frames of type $\{t_1, t_2, \ldots, t_n\}$ with $|\mathcal{A} \cap \mathcal{B}| = b$. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ with $|G_i| = t_i$, $1 \le i \le n$ and let Y be a set of 3 new points such that $X \cap Y = \emptyset$.

For $1 \le i \le n$, we have two RGDDs of type $3^{1+t_i/3}$ $(G_i \cup Y, \mathcal{H}_i, \mathcal{C}_i)$ and $(G_i \cup Y, \mathcal{H}_i, \mathcal{D}_i)$ with $|\mathcal{C}_i \cap \mathcal{D}_i| = b_i$ where $Y \in \mathcal{H}_i$. It is easy to see that $(X \cup Y, \mathcal{H}_n \cup (\bigcup_{1 \le i \le n-1} (\mathcal{H}_i \setminus \{Y\})), (\bigcup_{i=1}^{n} \mathcal{C}_i) \cup \mathcal{A})$ and $(X \cup Y, \mathcal{H}_n \cup (\bigcup_{1 \le i \le n-1} (\mathcal{H}_i \setminus \{Y\})), (\bigcup_{i=1}^{n} \mathcal{D}_i) \cup \mathcal{B})$ are two 3-RGDDs of type $3^{1+\sum_{i=1}^{n} t_i/3}$. (This is in fact the Filling in Holes Construction; see [10].) Clearly, the two newly 3-RGDDs have

$$|\mathcal{A} \cap \mathcal{B}| + \left| \left(\bigcup_{i=1}^{n} \mathcal{C}_i \right) \cap \left(\bigcup_{i=1}^{n} \mathcal{D}_i \right) \right| = b + \sum_{i=1}^{n} b_i$$

common triples. □

Theorem 2.3. *Suppose that $(X, \mathcal{G}, \mathcal{A})$ is a GDD, and w is a function from X to $Z^+ \cup \{0\}$. Let $f \in X$ be a fixed point and G_f the group containing the point f. For every block $A \in \mathcal{A}$ containing f, suppose that there is a pair of k-frames of type $\{w(x) : x \in A\}$ with a common holey parallel class corresponding to hole of size $w(f)$ intersecting in b_A other blocks; and for every block $A \in \mathcal{A}$ that does not contain f, suppose that there is a pair of k-frames of type $\{w(x) : x \in A\}$ with b_A common blocks. Then there exists a pair of k-frames of type $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$ with a common holey parallel class corresponding to hole of size $\sum_{x \in G_f} w(x)$ intersecting in $\sum_{A \in \mathcal{A}} b_A$ other blocks.*

Proof. For every $x \in X$, let Y_x be a set of cardinality $w(x)$, and for any $Z \subseteq X$, define $Y_Z = \bigcup_{x \in Z} Y_x$. For every block $A \in \mathcal{A}$ containing f, we have a pair of k-frames $(Y_A, \{Y_x : x \in A\}, \mathcal{B}_A)$ and $(Y_A, \{Y_x : x \in A\}, \mathcal{C}_A)$ with a common holey parallel class $P_A(Y_f)$ corresponding to hole Y_f, and $|\mathcal{B}_A \cap \mathcal{C}_A| = b_A + r_A$, r_A of them being the blocks of the common holey parallel class. Further, for every block $A \in \mathcal{A}$ that does not contain f, we have a pair of k-frames $(Y_A, \{Y_x : x \in A\}, \mathcal{B}_A)$ and $(Y_A, \{Y_x : x \in A\}, \mathcal{C}_A)$ with $|\mathcal{B}_A \cap \mathcal{C}_A| = b_A$. Then it is easy to see that $(Y_X, \{Y_G : G \in \mathcal{G}\}, \bigcup_{A \in \mathcal{A}} \mathcal{B}_A)$ and $(Y_X, \{Y_G : G \in \mathcal{G}\}, \bigcup_{A \in \mathcal{A}} \mathcal{C}_A)$ are k-frames (this is in fact the Fundamental Frame Construction; see [10]). From the construction and the assumption, it is trivial to see that $|\mathcal{B}_A \cap \mathcal{C}_B| = 0$ if $A \ne B$, or b_A if $f \notin A = B$, or $b_A + r_A$ if $f \in A = B$. Then, the newly obtained k-frames have a common holey parallel class $\bigcup_{\{A \in \mathcal{A} : f \in A\}} P_A(Y_f)$ corresponding to hole Y_{G_f}, and they have

$$\left| \left(\bigcup_{A \in \mathcal{A}} \mathcal{B}_A \right) \cap \left(\bigcup_{A \in \mathcal{A}} \mathcal{C}_A \right) \right| = \sum_{A \in \mathcal{A}} b_A + \sum_{\{A \in \mathcal{A} : f \in A\}} r_A$$

common blocks, $\sum_{\{A \in \mathcal{A} : f \in A\}} r_A$ of them being the blocks of the common holey parallel class. This completes the proof. □

Theorem 2.4. *Let $a \ge 0$ and suppose that there are two Kirkman frames of type $\{t_1, t_2, \ldots, t_n\}$ with a common holey parallel class P_1 with the hole corresponding to the n-th group, which intersect in b other triples. For $1 \le i \le n - 1$, suppose that* *there exists a pair of* KTS$(t_i + a)$s *containing the same sub-*KTS(a) *with* b_i *common triples, and suppose there exists a pair of* KTS$(t_n + a)$s *with a common parallel class* P_2 *intersecting in* b_n *other triples. Then, there exists a pair of* KTS$(\sum_{1\le i\le n} t_i + a)$s *with a common parallel class* $P_1 \cup P_2$ *intersecting in* $b + \sum_{1\le i\le n} b_i - (n-1)a(a-1)/6$ *other triples. Furthermore, we have* $b + \sum_{1\le i\le n} b_i - (n-1)a(a-1)/6 \in J_1[(\sum_{1\le i\le n} t_i + a)/3]$.

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ and $(X, \mathcal{G}, \mathcal{B})$ be two Kirkman frames of type $\{t_1, t_2, \dots, t_n\}$ with $|\mathcal{A} \cap \mathcal{B}| = b + r_1$, r_1 of them being the triples of the common holey parallel class P_1. Let $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$ with $|G_i| = t_i$, $1 \le i \le n$, and let Y be a set of cardinality a such that $X \cap Y = \emptyset$.

For $1 \le i \le n-1$, we have two KTS$(t_i + a)$s $(G_i \cup Y, \mathcal{C}_i)$ and $(G_i \cup Y, \mathcal{D}_i)$ containing the same sub-KTS(a) $(Y, \mathcal{E}_i)$, and with $|\mathcal{C}_i \cap \mathcal{D}_i| = b_i$. By the assumption, we also have two KTS$(t_n + a)$s $(G_n \cup Y, \mathcal{C}_n)$ and $(G_n \cup Y, \mathcal{D}_n)$ with a common parallel class P_2, and $|\mathcal{C}_n \cap \mathcal{D}_n| = b_n + r_2$, r_2 of them being the triples of the common parallel class. It is easy to see that $(X \cup Y, \mathcal{A} \cup (\bigcup_{1\le i\le n-1}(\mathcal{C}_i \setminus \mathcal{E}_i)) \cup \mathcal{C}_n)$ and $(X \cup Y, \mathcal{B} \cup (\bigcup_{1\le i\le n-1}(\mathcal{D}_i \setminus \mathcal{E}_i)) \cup \mathcal{D}_n)$ are two KTS$(\sum_{1\le i\le n} t_i + a)$s with a common parallel class $P_1 \cup P_2$ (this is in fact the Filling in Holes Construction; see [10]). Clearly, the two newly KTSs have

$$|\mathcal{A} \cap \mathcal{B}| + \sum_{1\le i\le n-1} |(\mathcal{C}_i \setminus \mathcal{E}_i) \cap (\mathcal{D}_i \setminus \mathcal{E}_i)| + |\mathcal{C}_n \cap \mathcal{D}_n| = b + \sum_{1\le i\le n} b_i - (n-1)a(a-1)/6 + (r_1 + r_2)$$

common triples, $r_1 + r_2$ of them being the triples of the common parallel class. This completes the proof. □

3. The case of $u = 3, 5, 7$

Lemma 3.1. $J_1[3] = \{0, 3, 9\}$.

Proof. Let $I_9 = \{0, 1, \dots, 8\}$. A KTS(9) $(I_9, \mathcal{A})$ is listed as

$\mathcal{A}$:	012	345	678	036	147	258
	048	237	156	057	138	246

Consider the permutations on I_9: $\pi_1 = (6\,8\,7)$ and $\pi_2 = (7\,8)$. It is readily checked that

$|\mathcal{A} \cap \pi_1\mathcal{A}| = 3$, the common parallel class is $\{0, 1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}$ and hence $0 \in J_1[3]$;
$|\mathcal{A} \cap \pi_2\mathcal{A}| = 6$, the common parallel class is $\{0, 1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}$ and hence $3 \in J_1[3]$;
$|\mathcal{A} \cap \mathcal{A}| = 12$, the common parallel class is $\{0, 1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}$ and hence $9 \in J_1[3]$.

It is well known that there exists only one KTS(9) under isomorphism. Denoted by Sym(I_9) all the permutations on I_9, we choose all the suitable $\pi \in$ Sym(I_9) which can make $\mathcal{A}$ and $\pi\mathcal{A}$ have a common parallel class. Then we find out the intersection number k between $\mathcal{A}$ and $\pi\mathcal{A}$ belongs to $\{3, 6, 12\}$. We obtain $J_1[3] \subseteq \{0, 3, 9\}$, and hence $J_1[3] = \{0, 3, 9\}$. □

Lemma 3.2. $J_1[5] = I_1[5] \setminus \{23, 24\}$.

Proof. Take the following seven non-isomorphic 3-RGDDs of type 3^5 on $I_{15} = \{0, 1, \dots, 14\}$ with group set $\mathcal{G}$: 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 (see [9]).

Block set $\mathcal{A}_1$:					
0 3 6	0 9 12	0 7 10	0 4 13	0 8 14	0 5 11
1 5 14	1 4 7	1 6 12	1 8 11	1 10 13	1 3 9
2 4 10	2 11 14	2 5 8	2 6 9	2 3 12	2 7 13
7 11 12	3 8 10	3 11 13	3 7 14	4 6 11	4 8 12
8 9 13	5 6 13	4 9 14	5 10 12	5 7 9	6 10 14
Block set $\mathcal{A}_2$:					
0 3 6	0 9 12	0 7 13	0 4 10	0 8 11	0 5 14
1 5 8	1 11 14	1 3 9	1 6 12	1 4 13	1 7 10
2 10 13	2 4 7	2 5 11	2 8 14	2 6 9	2 3 12
4 9 14	3 8 10	4 8 12	3 11 13	3 7 14	4 6 11
7 11 12	5 6 13	6 10 14	5 7 9	5 10 12	8 9 13
Block set $\mathcal{A}_3$:					
0 3 6	0 9 12	0 7 10	0 4 13	0 8 14	0 5 11
1 10 13	1 4 7	1 8 11	1 5 14	1 3 9	1 6 12
2 7 14	2 5 10	2 3 12	2 6 9	2 11 13	2 4 8
4 11 12	3 8 13	4 9 14	3 7 11	4 6 10	3 10 14
5 8 9	6 11 14	5 6 13	8 10 12	5 7 12	7 9 13

Block set $\mathcal{A}_4$:

0 3 6	0 9 12	0 4 13	0 5 7	0 10 14	0 8 11
1 4 10	1 7 13	1 6 12	1 11 14	1 5 8	1 3 9
2 7 14	2 4 11	2 8 10	2 6 9	2 3 12	2 5 13
5 11 12	3 8 14	3 7 11	3 10 13	4 7 9	4 6 14
8 9 13	5 6 10	5 9 14	4 8 12	6 11 13	7 10 12

Block set $\mathcal{A}_5$:

0 3 6	0 4 9	0 5 12	0 10 13	0 7 14	0 8 11
1 9 14	1 7 12	1 6 13	1 5 8	1 4 11	1 3 10
2 11 12	2 6 10	2 8 9	2 4 7	2 3 13	2 5 14
4 8 13	3 8 14	3 7 11	3 9 12	5 6 9	4 6 12
5 7 10	5 11 13	4 10 14	6 11 14	8 10 12	7 9 13

Block set $\mathcal{A}_6$:

0 3 6	0 9 12	0 4 13	0 7 10	0 11 14	0 5 8
1 7 13	1 4 10	1 5 14	1 8 11	1 3 9	1 6 12
2 4 11	2 8 13	2 6 9	2 3 12	2 5 7	2 10 14
5 10 12	3 7 14	3 8 10	4 6 14	4 8 12	3 11 13
8 9 14	5 6 11	7 11 12	5 9 13	6 10 13	4 7 9

Block set $\mathcal{A}_7$:

0 3 6	0 9 12	0 4 7	0 5 13	0 8 10	0 11 14
1 11 13	1 5 10	1 6 12	1 7 14	1 3 9	1 4 8
2 7 10	2 8 14	2 5 11	2 6 9	2 4 13	2 3 12
4 9 14	3 7 13	3 10 14	3 8 11	5 6 14	5 7 9
5 8 12	4 6 11	8 9 13	4 10 12	7 11 12	6 10 13

Consider the following permutations on I_{15} which fix the group set $\mathcal{G}$:

$$\pi_0 = (6\,7\,8)(9\,11\,10)(12\,13\,14), \qquad \pi_1 = (7\,8)(9\,11\,10)(12\,13\,14),$$
$$\pi_2 = (6\,7)(9\,10\,11)(12\,13\,14), \qquad \pi_3 = (9\,10\,11)(12\,13\,14),$$
$$\pi_4 = (4\,5)(6\,7)(10\,11)(12\,14), \qquad \pi_5 = (7\,8)(10\,11)(12\,14),$$
$$\pi_6 = (10\,11)(12\,13\,14), \qquad \pi_7 = (9\,10\,11)(12\,14),$$
$$\pi_8 = (3\,4)(6\,7)(9\,10)(12\,13), \qquad \pi_9 = (1\,2)(7\,8)(10\,11),$$
$$\pi_{10} = (10\,11)(13\,14), \qquad \pi_{11} = (0\,1\,2)(3\,4)(6\,7\,8)(9\,10\,11)(12\,13\,14),$$
$$\pi_{12} = (12\,13\,14), \qquad \pi_{13} = (0\,1\,2)(3\,4)(6\,7\,8)(9\,10\,11)(12\,14\,13),$$
$$\pi_{14} = (0\,1\,2)(3\,4)(6\,7\,8)(9\,10\,11)(12\,13), \qquad \pi_{15} = (4\,5)(7\,8)(13\,14),$$
$$\pi_{16} = (1), \qquad \pi_{17} = (9\,10)(12\,14\,13),$$
$$\pi_{18} = (13\,14), \qquad \pi_{19} = (6\,10\,8\,11\,7\,9)(12\,13),$$
$$\pi_{20} = (7\,8)(10\,11), \qquad \pi_{21} = (4\,5)(10\,11),$$
$$\pi_{22} = (4\,5)(7\,8)(10\,11)(13\,14), \qquad \pi_{26} = (0\,1)(4\,5)(7\,8)(10\,11)(13\,14),$$
$$\pi_{30} = (1).$$

It is readily checked that for each row in Table 1, $|\mathcal{A}_i \cap \pi_k \mathcal{A}_j| = k$. □

There are only seven non-isomorphic 3-RGDD $(I_{15}, \mathcal{G}, \mathcal{A}_i)$ $(i = 1, 2, \ldots, 7)$ of type 3^5 (see [9]). The computer has searched for all the permutations on I_{15} which can keep group set $\mathcal{G}$ fixed. Let $\mathcal{F}$ denote the obtained permutations. Then make a permutation $\pi \in \mathcal{F}$ on $\mathcal{A}_i$ $(i = 1, 2, \ldots, 7)$, we obtain 3-RGGDs of type 3^5 which have the same group set $\mathcal{G}$. Then we find out the intersection numbers between two of the obtained RGDDs belong to $I_1[5] \setminus \{23, 24\}$. That is to say $J_1[5] \subseteq I_1[5] \setminus \{23, 24\}$. Hence, $J_1[5] = I_1[5] \setminus \{23, 24\}$. □

Lemma 3.3. $\{0 - 27, 31, 33, 36, 45, 63\} \subseteq J_1[7]$.

Proof. Let $I_{21} = \{0, 1, \ldots, 20\}$, group set $\mathcal{G} = \{\{3j, 3j+1, 3j+2\} : 0 \le j \le 6\}$ and $\mathcal{A}$ contains the following triples:

0 3 12	1 13 20	2 11 17	4 6 19	5 14 18	7 10 16	8 9 15
0 6 9	1 14 15	2 4 20	3 10 18	5 8 19	7 13 17	11 12 16
0 11 13	1 8 17	2 15 19	3 14 16	4 12 18	5 7 9	6 10 20
0 7 20	1 10 19	2 3 9	4 13 16	5 12 15	6 14 17	8 11 18
0 15 18	1 5 16	2 6 12	3 7 11	4 10 17	8 14 20	9 13 19
0 10 14	1 4 11	2 8 16	3 15 20	5 6 13	7 12 19	9 17 18
0 16 19	1 7 18	2 5 10	3 8 13	4 9 14	6 11 15	12 17 20
0 5 17	1 3 6	2 13 18	4 7 15	8 10 12	9 16 20	11 14 19
0 4 8	1 9 12	2 7 14	3 17 19	5 11 20	6 16 18	10 13 15

Table 1
Intersection numbers for Lemma 3.2.

i	j	k
1	1	0, . . . , 6, 10, 12, 18, 30
3	3	7, 8, 9, 11, 13, 14
3	4	15
1	3	16
4	5	17
5	7	19
2	3	20
4	7	21
3	6	22, 26

Then $(I_{21}, \mathcal{G}, \mathcal{A})$ is a 3-RGDD of type 3^7. Consider the following permutations on I_{21} which fix the group set $\mathcal{G}$:

$\pi_0 = (6\ 7\ 8)(9\ 10\ 11)(12\ 13)(15\ 17\ 16)(18\ 19\ 20)$, $\pi_1 = (7\ 8)(9\ 10\ 11)(12\ 13)(15\ 17\ 16)(18\ 19\ 20)$,
$\pi_2 = (9\ 10\ 11)(12\ 13\ 14)(15\ 16\ 17)(18\ 20\ 19)$, $\pi_3 = (9\ 10\ 11)(12\ 13)(15\ 17\ 16)(18\ 19\ 20)$,
$\pi_4 = (9\ 10)(12\ 13\ 14)(15\ 16\ 17)(18\ 20\ 19)$, $\pi_5 = (9\ 10)(12\ 13)(15\ 16\ 17)(18\ 20\ 19)$,
$\pi_6 = (10\ 11)(12\ 13\ 14)(15\ 16\ 17)(18\ 20\ 19)$, $\pi_7 = (10\ 11)(12\ 13\ 14)(15\ 17\ 16)(18\ 19\ 20)$,
$\pi_8 = (12\ 13\ 14)(15\ 16\ 17)(18\ 20\ 19)$, $\pi_9 = (10\ 11)(12\ 13\ 14)(16\ 17)(18\ 19\ 20)$,
$\pi_{10} = (10\ 11)(13\ 14)(15\ 16\ 17)(18\ 20\ 19)$, $\pi_{11} = (12\ 13)(15\ 16\ 17)(18\ 20\ 19)$,
$\pi_{12} = (13\ 14)(15\ 16\ 17)(18\ 20\ 19)$, $\pi_{13} = (12\ 13\ 14)(15\ 16)(18\ 19\ 20)$,
$\pi_{14} = (12\ 13)(15\ 16\ 17)(18\ 19\ 20)$, $\pi_{15} = (13\ 14)(15\ 16\ 17)(18\ 19\ 20)$,
$\pi_{16} = (12\ 13)(16\ 17)(18\ 19\ 20)$, $\pi_{17} = (13\ 14)(16\ 17)(18\ 19\ 20)$,
$\pi_{18} = (15\ 16\ 17)(18\ 20\ 19)$, $\pi_{19} = (10\ 11)(13\ 14)(16\ 17)(18\ 20)$,
$\pi_{20} = (12\ 13)(16\ 17)(19\ 20)$, $\pi_{21} = (15\ 16\ 17)(18\ 19\ 20)$,
$\pi_{22} = (12\ 13)(16\ 17)(18\ 20)$, $\pi_{23} = (13\ 14)(16\ 17)(18\ 20)$,
$\pi_{24} = (13\ 14)(18\ 19\ 20)$, $\pi_{25} = (16\ 17)(18\ 19\ 20)$,
$\pi_{26} = (9\ 10\ 11)(13\ 14)$, $\pi_{27} = (3\ 5)(6\ 8)(9\ 11)$,
$\pi_{31} = (16\ 17)(19\ 20)$, $\pi_{33} = (16\ 17)(18\ 20)$,
$\pi_{36} = (18\ 19\ 20)$, $\pi_{45} = (19\ 20)$,
$\pi_{63} = (1)$.

It is readily checked that for each $k \in \{0-27, 31, 33, 36, 45, 63\}$, $|\mathcal{A} \cap \pi_k\mathcal{A}| = k$. $\square$

Lemma 3.4. $28, 29, 30, 32, 37, 40, 46, 47, 49, 51, 53, 55 \in J_1[7]$.

Proof. Let $I_{21} = \{0, 1, \ldots, 20\}$, group set $\mathcal{G} = \{\{3j, 3j+1, 3j+2\} : 0 \le j \le 6\}$ and block set $\mathcal{A}$ contains the following triples:

0 8 15	1 12 19	2 3 7	4 6 9	5 16 20	10 13 17	11 14 18
0 7 18	1 13 15	2 9 16	3 10 14	4 17 20	5 6 12	8 11 19
0 11 16	1 9 20	2 6 17	3 8 13	4 15 19	5 7 14	10 12 18
0 10 19	1 3 11	2 4 13	5 15 18	6 14 16	7 12 20	8 9 17
0 13 20	1 4 14	2 5 10	3 17 18	6 11 15	7 9 19	8 12 16
0 14 17	1 5 8	2 11 20	3 9 12	4 16 18	6 13 19	7 10 15
0 3 6	1 10 16	2 12 15	4 7 11	5 17 19	8 14 20	9 13 18
0 5 9	1 6 18	2 14 19	3 15 20	4 8 10	7 13 16	11 12 17
0 4 12	1 7 17	2 8 18	3 16 19	5 11 13	6 10 20	9 14 15

block set $\mathcal{B}$ contains the following triples:

0 10 19	5 17 20	4 7 11	9 13 18	1 3 8	6 14 16	2 12 15
0 11 16	5 6 12	4 17 18	7 9 19	2 3 10	8 14 20	1 13 15
0 7 18	5 10 14	2 4 13	1 9 20	3 17 19	6 11 15	8 12 16
0 8 15	2 5 7	4 6 9	3 16 20	1 12 19	11 14 18	10 13 17
0 14 17	5 15 19	4 8 10	2 9 16	3 11 13	1 6 18	7 12 20
0 13 20	5 16 18	1 4 14	3 9 12	2 6 17	8 11 19	7 10 15
0 5 9	4 15 20	3 7 14	6 13 19	11 12 17	2 8 18	1 10 16
0 3 6	5 8 13	4 16 19	9 14 15	10 12 18	2 11 20	1 7 17
0 4 12	1 5 11	8 9 17	3 15 18	6 10 20	2 14 19	7 13 16

Then $(I_{21}, \mathcal{G}, \mathcal{A})$ and $(I_{21}, \mathcal{G}, \mathcal{B})$ are two 3-RGDDs of type 3^7. Consider the following permutations on I_{21} which fix the group set $\mathcal{G}$:

$$\begin{aligned}
\pi_{28} &= (1\ 2)(3\ 5)(6\ 7\ 8)(9\ 10\ 11)(12\ 14\ 13)(15\ 17)(19\ 20),\\
\pi_{29} &= (1\ 2)(3\ 5)(6\ 7\ 8)(9\ 11\ 10)(13\ 14)(15\ 17)(19\ 20),\\
\pi_{30} &= (0\ 1)(3\ 5)(6\ 7\ 8)(9\ 11\ 10)(13\ 14)(15\ 17)(19\ 20),\\
\pi_{32} &= (0\ 1)(3\ 5)(6\ 7\ 8)(9\ 10\ 11)(12\ 14\ 13)(15\ 17)(19\ 20),\\
\pi_{37} &= (0\ 1\ 2)(3\ 5)(6\ 7\ 8)(9\ 11\ 10)(12\ 14\ 13)(15\ 17)(19\ 20),\\
\pi_{40} &= (0\ 6\ 12\ 9)(1\ 8\ 13\ 11)(2\ 7\ 14\ 10)(3\ 5)(15\ 19\ 17\ 20\ 16\ 18),\\
\pi_{47} &= (0\ 7\ 11)(1\ 8\ 9)(2\ 6\ 10)(3\ 15\ 20)(4\ 16\ 18)(5\ 17\ 19)(12\ 13\ 14),\\
\pi_{49} &= (0\ 6)(1\ 7)(2\ 8)(3\ 5)(9\ 12)(10\ 13)(11\ 14)(15\ 17),\\
\pi_{51} &= (0\ 7)(1\ 6)(2\ 8)(3\ 15)(4\ 16)(5\ 17)(9\ 10)(12\ 13),\\
\pi_{53} &= (0\ 6\ 12\ 9)(1\ 8\ 13\ 11)(2\ 7\ 14\ 10)(3\ 5)(15\ 19\ 17\ 20)(16\ 18),\\
\pi_{55} &= (0\ 6\ 12\ 9)(1\ 8\ 13\ 11)(2\ 7\ 14\ 10)(3\ 5)(15\ 19)(16\ 18)(17\ 20).
\end{aligned}$$

It is readily checked that for each $k \in \{28, 29, 30, 32, 37, 40, 47, 49, 51, 53, 55\}$, $|\mathcal{A} \cap \pi_k\mathcal{A}| = k$ and $|\mathcal{A} \cap \mathcal{B}| = 46$. □

Lemma 3.5. $34, 38, 39, 41, 43, 44, 56, 57, 59 \in J_1[7]$.

Proof. Let $I_{21} = \{0, 1, \dots, 20\}$, group set $\mathcal{G} = \{\{3j, 3j+1, 3j+2\} : 0 \le j \le 6\}$ and $\mathcal{A}$ contains the following triples:

0 4 17	1 5 6	2 9 19	3 7 11	8 12 16	10 14 18	13 15 20
0 14 16	1 8 15	2 6 10	3 17 19	4 11 12	5 13 18	7 9 20
0 5 7	1 11 18	2 8 14	3 16 20	4 13 19	6 9 15	10 12 17
0 3 9	1 10 13	2 16 18	4 6 20	5 12 19	7 14 15	8 11 17
0 12 15	1 3 14	2 11 20	4 8 18	5 9 16	6 13 17	7 10 19
0 6 18	1 12 20	2 5 17	3 8 10	4 9 14	7 13 16	11 15 19
0 10 20	1 16 19	2 4 15	3 6 12	5 11 14	7 17 18	8 9 13
0 11 13	1 9 17	2 7 12	3 15 18	4 10 16	5 8 20	6 14 19
0 8 19	1 4 7	2 3 13	5 10 15	6 11 16	9 12 18	14 17 20

Then $(I_{21}, \mathcal{G}, \mathcal{A})$ is a 3-RGDD of type 3^7. Consider the following permutations on I_{21} which fix the group set $\mathcal{G}$:

$$\begin{aligned}
\pi_{34} &= (0\ 12\ 1\ 14\ 2\ 13)(4\ 5)(6\ 9)(7\ 11)(8\ 10)(15\ 18)(16\ 20)(17\ 19),\\
\pi_{38} &= (0\ 15\ 9\ 12\ 18\ 6)(1\ 17\ 10\ 14\ 19\ 8)(2\ 16\ 11\ 13\ 20\ 7),\\
\pi_{39} &= (0\ 9\ 18)(1\ 10\ 19)(2\ 11\ 20)(6\ 15\ 12)(7\ 16\ 13\ 8\ 17\ 14),\\
\pi_{41} &= (0\ 9\ 18)(1\ 10\ 19)(2\ 11\ 20)(6\ 15\ 12\ 8\ 17\ 14)(7\ 16\ 13),\\
\pi_{43} &= (0\ 19\ 11)(1\ 20\ 9)(2\ 18\ 10)(3\ 4\ 5)(6\ 16\ 14)(7\ 17\ 12)(8\ 15\ 13),\\
\pi_{44} &= (0\ 16\ 18\ 7\ 9\ 13)(1\ 15\ 19\ 6\ 10\ 12)(2\ 17\ 20\ 8\ 11\ 14)(3\ 4),\\
\pi_{56} &= (0\ 15\ 9\ 12\ 18\ 6)(1\ 17\ 10\ 14\ 19\ 8)(2\ 16\ 11\ 13\ 20\ 7)(4\ 5),\\
\pi_{57} &= (0\ 9\ 18)(1\ 10\ 19)(2\ 11\ 20)(6\ 15\ 12)(7\ 16\ 13)(8\ 17\ 14),\\
\pi_{59} &= (0\ 12)(1\ 14)(2\ 13)(4\ 5)(6\ 9)(7\ 11)(8\ 10)(15\ 18)(16\ 20)(17\ 19).
\end{aligned}$$

It is readily checked that for each $k \in \{34, 38, 39, 41, 43, 44, 56, 57, 59\}$, $|\mathcal{A} \cap \pi_k\mathcal{A}| = k$. □

Lemma 3.6. $35 \in J_1[7]$.

Proof. Let $I_{21} = \{0, 1, \dots, 20\}$. A 3-RGDD $(I_{21}, \mathcal{G}, \mathcal{A})$ of type 3^7 is listed as follows: Group set $\mathcal{G} = \{\{3j, 3j+1, 3j+2\} : 0 \le j \le 6\}$

Block set $\mathcal{A}$:

0 15 18	1 13 17	2 3 10	4 6 9	5 16 20	7 11 12	8 14 19
0 11 16	1 9 19	2 4 13	3 8 15	5 7 14	6 17 20	10 12 18
0 10 20	1 7 16	2 14 18	3 9 12	4 8 17	5 11 13	6 15 19
0 14 17	1 3 20	2 7 9	4 11 15	5 8 10	6 13 18	12 16 19
0 8 13	1 10 15	2 11 19	3 14 16	4 7 20	5 6 12	9 17 18
0 7 19	1 8 12	2 5 15	3 11 17	4 16 18	6 10 14	9 13 20
0 3 6	1 5 18	2 12 17	4 10 19	7 13 15	8 9 16	11 14 20
0 5 9	1 4 14	2 6 16	3 13 19	7 10 17	8 11 18	12 15 20
0 4 12	1 6 11	2 8 20	3 7 18	5 17 19	9 14 15	10 13 16

The permutation $\pi = (3\ 4)(15\ 17)$ fixes the group set $\mathcal{G}$ and it is readily checked that $|\mathcal{A} \cap \pi\mathcal{A}| = 35$. □

Lemma 3.7. $48 \in J_1[7]$.

Proof. Let $I_{21} = \{0, 1, \ldots, 20\}$, group set $\mathcal{G} = \{\{3j, 3j+1, 3j+2\} : 0 \le j \le 6\}$ and block set $\mathcal{A}$ contains the following triples:

0 15 19	3 9 12	1 6 14	4 11 13	5 17 18	7 10 16	2 8 20
0 13 20	6 16 19	9 15 18	3 10 14	1 4 8	11 12 17	2 5 7
0 7 18	6 17 20	9 14 16	3 8 13	2 4 15	10 12 19	1 5 11
0 11 16	2 6 10	8 9 17	3 15 20	4 7 14	1 12 18	5 13 19
0 14 17	4 6 9	1 3 19	7 12 20	5 8 16	10 13 15	2 11 18
0 8 10	5 6 12	2 9 13	3 16 18	4 17 19	1 7 15	11 14 20
0 3 6	1 9 20	4 10 18	2 12 16	5 14 15	7 13 17	8 11 19
0 4 12	6 11 15	7 9 19	2 3 17	5 10 20	1 13 16	8 14 18
0 5 9	6 13 18	3 7 11	4 16 20	8 12 15	1 10 17	2 14 19

block set $\mathcal{B}$ contains the following triples:

5 7 19	1 13 16	8 10 12	0 14 17	3 15 20	2 11 18	4 6 9
4 7 14	5 10 18	1 3 19	2 12 16	8 9 17	0 13 20	6 11 15
3 7 11	4 10 20	1 12 18	9 14 16	6 17 19	2 5 13	0 8 15
0 7 18	2 6 10	1 9 20	4 16 19	11 12 17	3 8 13	5 14 15
7 12 20	1 10 17	5 8 16	4 11 13	9 15 18	2 14 19	0 3 6
2 7 9	10 13 15	1 6 14	3 16 18	5 17 20	8 11 19	0 4 12
7 10 16	1 4 8	2 3 17	6 13 18	12 15 19	11 14 20	0 5 9
7 13 17	0 10 19	1 5 11	6 16 20	2 4 15	8 14 18	3 9 12
1 7 15	3 10 14	0 11 16	4 17 18	9 13 19	2 8 20	5 6 12

Then $(I_{21}, \mathcal{G}, \mathcal{A})$ and $(I_{21}, \mathcal{G}, \mathcal{B})$ are two 3-RGDDs of type 3^7. It is readily checked that $|\mathcal{A} \cap \mathcal{B}| = 48$. □

Lemma 3.8. $50 \in J_1[7]$.

Proof. Let $I_{21} = \{0, 1, \ldots, 20\}$, group set $\mathcal{G} = \{\{3j, 3j+1, 3j+2\} : 0 \le j \le 6\}$ and block set $\mathcal{A}$ contains the following triples:

0 5 7	6 9 15	3 11 13	1 12 17	8 16 18	4 10 19	2 14 20
0 13 20	9 12 18	3 14 19	6 10 17	1 11 15	4 7 16	2 5 8
0 8 19	2 3 16	4 6 11	1 9 14	7 12 20	5 10 15	13 17 18
0 4 17	3 10 20	6 14 16	2 7 9	11 12 19	8 13 15	1 5 18
0 11 16	3 6 12	5 9 13	2 15 19	4 14 18	1 7 10	8 17 20
0 10 14	3 15 18	1 6 20	4 8 9	5 12 16	7 13 19	2 11 17
0 3 9	2 6 13	8 10 12	4 15 20	7 11 18	1 16 19	5 14 17
0 12 15	3 7 17	5 6 19	9 16 20	2 10 18	1 4 13	8 11 14
0 6 18	1 3 8	9 17 19	2 4 12	7 14 15	10 13 16	5 11 20

block set $\mathcal{B}$ contains the following triples:

0 5 19	2 11 17	8 13 15	4 14 18	9 16 20	3 6 12	1 7 10
5 7 15	2 14 20	8 10 12	4 6 11	13 17 18	0 3 9	1 16 19
1 5 18	3 8 19	0 11 16	2 6 13	10 14 15	4 9 17	7 12 20
5 12 16	1 8 9	3 11 13	2 10 18	0 7 14	6 17 19	4 15 20
5 9 13	8 11 14	2 15 19	1 12 17	3 10 20	0 6 18	4 7 16
5 6 10	8 17 20	7 11 18	2 3 16	9 14 19	0 12 15	1 4 13
2 5 8	1 11 15	6 14 16	3 7 17	0 13 20	9 12 18	4 10 19
5 14 17	0 4 8	11 12 19	2 7 9	1 6 20	3 15 18	10 13 16
5 11 20	8 16 18	2 4 12	1 3 14	0 10 17	6 9 15	7 13 19

Then $(I_{21}, \mathcal{G}, \mathcal{A})$ and $(I_{21}, \mathcal{G}, \mathcal{B})$ are two 3-RGDDs of type 3^7. It is readily checked that $|\mathcal{A} \cap \mathcal{B}| = 50$. □

Lemma 3.9. $52 \in J_1[7]$.

Proof. Let $I_{21} = \{0, 1, \ldots, 20\}$, group set $\mathcal{G} = \{\{3j, 3j+1, 3j+2\} : 0 \le j \le 6\}$ and block set $\mathcal{A}$ contains the following triples:

0 3 9	6 17 19	5 12 16	2 13 15	1 8 18	4 7 10	11 14 20
3 6 12	0 10 20	8 9 16	1 11 15	2 7 18	4 13 19	5 14 17
6 9 15	0 14 16	3 11 13	4 8 12	10 17 18	1 7 19	2 5 20
9 12 18	0 5 19	3 8 10	6 13 20	7 14 15	1 4 16	2 11 17
0 12 15	3 7 17	4 6 11	1 9 20	5 13 18	10 16 19	2 8 14
3 15 18	0 4 17	2 6 16	9 14 19	7 12 20	1 10 13	5 8 11
0 6 18	1 3 14	2 4 9	11 12 19	5 10 15	7 13 16	8 17 20
0 8 13	2 3 19	6 10 14	5 7 9	1 12 17	4 15 20	11 16 18
0 7 11	3 16 20	1 5 6	9 13 17	2 10 12	8 15 19	4 14 18

block set $\mathcal{B}$ contains the following triples:

0 9 12	1 3 14	10 17 18	2 6 16	4 15 20	7 13 19	5 8 11
3 9 18	0 7 11	1 12 17	6 13 20	5 10 15	4 16 19	2 8 14
3 6 12	0 10 20	5 7 9	4 14 18	8 15 19	1 13 16	2 11 17
12 15 18	0 8 13	1 9 20	2 3 19	4 6 11	7 10 16	5 14 17
0 6 18	2 4 9	3 11 13	5 12 16	7 14 15	1 10 19	8 17 20
6 9 15	0 14 16	3 7 17	11 12 19	1 8 18	4 10 13	2 5 20
0 3 15	8 9 16	2 10 12	5 13 18	6 17 19	1 4 7	11 14 20
0 5 19	9 13 17	3 16 20	4 8 12	2 7 18	6 10 14	1 11 15
0 4 17	9 14 19	3 8 10	7 12 20	11 16 18	1 5 6	2 13 15

Then $(I_{21}, \mathcal{G}, \mathcal{A})$ and $(I_{21}, \mathcal{G}, \mathcal{B})$ are two 3-RGDDs of type 3^7. It is readily checked that $|\mathcal{A} \cap \mathcal{B}| = 52$. □

Lemma 3.10. $42, 54 \in J_1[7]$.

Proof. By Lemma 10 in [6], $(S, \mathcal{B})$ and $(S, \mathcal{C})$ are KTS(21)s with a common parallel class and $|\mathcal{B} \cap \mathcal{C}| = 49$, so we obtain $42 \in J_1[7]$; by Lemma 3.13 in [3], $(I_{21}, \mathcal{D}_1)$ and $(I_{21}, \mathcal{D}_2)$ are KTS(21)s with a common parallel class and $|\mathcal{D}_1 \cap \mathcal{D}_2| = 61$, so we obtain $54 \in J_1[7]$. □

Lemma 3.11. $J_1[7] = I_1[7]$.

Proof. It follows by Lemmas 3.3–3.10. □

4. Working lemmas

In this section, for any positive integer n, we always denote by $S(n)$ the set of all non-negative integers less than or equal to n, with the exceptions of $n-5$, $n-3$, $n-2$ and $n-1$. For a positive integer u, let $t_u = 3\binom{u}{2}$.

Lemma 4.1 *([1]). A* B$(\{4, 7^*\}, 1; v)$ *(with exactly one block of size 7) exists if and only if* $v \equiv 7, 10 \pmod{12}$, $v \ne 10, 19$.

Lemma 4.2 *([2, Lemma 4.1]). For any* $k \in \{0, 2, 8\}$, *there exists a pair of Kirkman frames of type* 2^4 *having* k *triples in common.*

Lemma 4.3 *([3, Theorem 3.6]). For any* $k \in \{0-21, 24, 28\}$, *there exists a pair of Kirkman frames with type* 2^7 *having* k *common triples.*

Lemma 4.4. *Let* $n = 3(u-1)(u-3)/2$. *For any* $k \in S(n)$, *there exists a pair of Kirkman frames of type* $6^{(u-1)/2}$ *having* k *common triples for* $u \equiv 5, 7 \pmod 8$ *and* $u \ge 15$.

Proof. Since $u \equiv 5, 7 \pmod 8$ and $u \ge 15$, by Lemma 4.1, there is a $((3u-1)/2, \{4, 7^*\}, 1)$-PBD, in which contains unique block of size 7. There is at least one point which does not occur in the block of size 7. Delete this point to get a $\{4, 7\}$-GDD with group type $3^{(u-1)/2}$, which contains unique block of size 7 and $[3(u-1)(u-3) - 56]/16 = (n-28)/8$ blocks of size 4. Give every point of the GDD weight 2. The required Kirkman frames of types 2^4 and 2^7 are guaranteed by Lemmas 4.2 and 4.3. Applying Theorem 2.1, we then obtain a pair of Kirkman frames having

$$2s + 8t + k' \tag{1}$$

common triples, where s, t are non-negative integers such that $0 \le s + t \le (n-28)/8$ and $k' \in \{0-21, 24, 28\}$. For any $k \in S(n)$, it is easy to see that k can be written in the form of (1). This completes the proof. □

Theorem 4.5. $J_1[u] = I_1[u]$ *for any* $u \equiv 5, 7 \pmod 8$ *and* $u \ge 15$.

Proof. By Lemma 3.1, there exists a pair of RGDDs of type 3^3 having k common triples where $k \in \{0, 3, 9\}$. Apply Theorem 2.2 and Lemma 4.4 to obtain a pair of 3-RGDDs of type 3^u where $u \equiv 5, 7 \pmod 8$ and $u \geq 15$ having

$$b + b_1 + b_2 + \cdots + b_{(u-1)/2} \tag{2}$$

common triples, where $b_i \in \{0, 3, 9\}$ for $1 \leq i \leq (u-1)/2$ and $b \in S(3(u-1)(u-3)/2)$. For any $k \in I_1[u]$, it is not difficult to show that k can be written in the form of (2). That is $I_1[u] \subseteq J_1[u]$ and hence $J_1[u] = I_1[u]$ for any $u \equiv 5, 7 \pmod 8$ and $u \geq 15$. □

Lemma 4.6 ([3, Lemma 4.6]). *For any $u \equiv 1, 3 \pmod 8$ and $u \geq 9$, there exists a pair of Kirkman frames of type $6^{(u-1)/2}$ having one triple in common.*

Lemma 4.7. *For any $k \in \{0, 1, 2, \ldots, 3\binom{u}{2} - 20\}$, there exists a pair of 3-RGDDs of type 3^u having k common triples for $u \equiv 1, 3 \pmod 8$ and $u \geq 9$.*

Proof. Since $u \equiv 1, 3 \pmod 8$ and $u \geq 9$, there exists a $((3u-1)/2, 4, 1)$-BIBD (see [4]). Delete a point of the BIBD to obtain a 4-GDD with group type $3^{(u-1)/2}$ which contains $n = 3(u-1)(u-3)/16$ blocks of size 4. Give each point of the GDD weight 2. Apply Theorem 2.1 and Lemma 4.2 to obtain a pair of Kirkman frame of group type $6^{(u-1)/2}$ with $2s + 8t$ triples in common, where s, t are non-negative integers such that $0 \leq s + t \leq n$. By Theorem 2.2, Lemmas 3.1 and 4.6, there is a pair of 3-RGDDs of type 3^u which intersect in

$$2s + 8t + \sum_{1 \leq i \leq (u-1)/2} b_i \tag{3}$$

or

$$1 + \sum_{1 \leq i \leq (u-1)/2} b_i \tag{4}$$

common triples, where s, t are non-negative integers such that $0 \leq s + t \leq n$ and $b_i \in \{0, 3, 9\}$ for $1 \leq i \leq (u-1)/2$. Note that $\{2s + 8t : s, t \text{ are non-negative integers}, 0 \leq s + t \leq n\} = \{2i : 0 \leq i \leq 4n\} \setminus \{8n-10, 8n-4, 8n-2\}$; and $\{\sum_{1 \leq i \leq (u-1)/2} b_i : b_i \in \{0, 3, 9\}, 1 \leq i \leq (u-1)/2\} = \{3j : 0 \leq j \leq 3(u-1)/2, j \neq 3(u-1)/2 - 1\}$. For any $k \in \{0, 1, 2, \ldots, 3\binom{u}{2} - 20\}$, it is not difficult to show that k can be written in the form of either (3) or (4). This completes the proof. □

Lemma 4.8. *For any $k \in \{9, 15, 21, 27, 33, 45, 81\}$, there exists a pair of 3-RGDDs of group type 9^3 with a common parallel class having k common triples.*

Proof. Let $I_{27} = \{0, 1, \ldots, 26\}$ and $\mathcal{G} = \{\{i + 9j : 0 \leq i \leq 8\} : j = 0, 1, 2\}$. A RGDD $(I_{27}, \mathcal{G}, \mathcal{A})$ of type 9^3 is constructed by listing the block set $\mathcal{A}$ as below:

0 9 18	1 10 19	2 11 20	3 12 21	4 13 22	5 14 23	6 15 24	7 16 25	8 17 26 (∗)
0 10 20	1 11 18	2 9 19	3 13 23	4 14 21	5 12 22	6 16 26	7 17 24	8 15 25
0 11 19	1 9 20	2 10 18	3 14 22	4 12 23	5 13 21	6 17 25	7 15 26	8 16 24
0 12 24	1 13 25	2 14 26	3 15 18	4 16 19	5 17 20	6 9 21	7 10 22	8 11 23
0 13 26	1 14 24	2 12 25	3 16 20	4 17 18	5 15 19	6 10 23	7 11 21	8 9 22
0 14 25	1 12 26	2 13 24	3 17 19	4 15 20	5 16 18	6 11 22	7 9 23	8 10 21
0 15 21	1 16 22	2 17 23	3 9 24	4 10 25	5 11 26	6 12 18	7 13 19	8 14 20
0 16 23	1 17 21	2 15 22	3 10 26	4 11 24	5 9 25	6 13 20	7 14 18	8 12 19
0 17 22	1 15 23	2 16 21	3 11 25	4 9 26	5 10 24	6 14 19	7 12 20	8 13 18

Consider the permutations on I_{27}:

$$\begin{aligned}
\pi_9 &= (0\ 3\ 7\ 2\ 6)(9\ 12\ 16\ 11\ 15)(18\ 21\ 25\ 20\ 24),\\
\pi_{15} &= (0\ 1\ 2\ 3)(9\ 10\ 11\ 12)(18\ 19\ 20\ 21),\\
\pi_{21} &= (3\ 4\ 7\ 8)(12\ 13\ 16\ 17)(21\ 22\ 25\ 26),\\
\pi_{27} &= (3\ 4\ 5)(12\ 13\ 14)(21\ 22\ 23),\\
\pi_{33} &= (2\ 3)(11\ 12)(20\ 21)(0\ 6\ 5\ 1)(9\ 15\ 14\ 10)(18\ 24\ 23\ 19),\\
\pi_{45} &= (0\ 1)(9\ 10)(18\ 19),\\
\pi_{81} &= (0).
\end{aligned}$$

For $k \in \{9, 15, 21, 27, 33, 45, 81\}$, it is readily checked that $|\mathcal{A} \cap \pi_k \mathcal{A}| = k$ with a common parallel class (∗). □

Lemma 4.9. $I_1[9] \setminus \{95, 101, 104\} \subseteq J_1[9]$.

Proof. By Lemma 4.8, there exists a pair of RGDDs of type 9^3 with a common parallel class which intersect in k triples where $k \in \{9, 15, 21, 27, 33, 45, 81\}$. Fill the ith $(1 \le i \le 3)$ group by a pair of KTS(9)s with b_i common triples where $b_i \in \{0, 1, 2, 3, 4, 6, 12\}$ from [2]. We then obtain a pair of KTS(27)s with a common parallel class which intersect in $k - 9 + \sum_{1\le i\le 3} b_i$ other common triples. Hence, $k - 9 + \sum_{1\le i\le 3} b_i \in J_1[9]$ where $k \in \{9, 15, 21, 27, 33, 45, 81\}$ and $b_i \in \{0, 1, 2, 3, 4, 6, 12\}$ for $1 \le i \le 3$. This implies that $I_1[9] \setminus \{59, 65, 67 - 71, 95, 101, 104\} \subseteq J_1[9]$. Together with $\{0, 1, \dots, 72\} \subseteq J_1[9]$ from Lemma 4.7, we have $I_1[9] \setminus \{95, 101, 104\} \subseteq J_1[9]$. □

Lemma 4.10. *There exists a pair of Kirkman frames of type* 2^4 *with a common holey parallel class having* k *other common triples where* $k \in \{0, 6\}$.

Proof. Let $X = \{1, 2, \dots, 8\}$ and $\mathcal{G} = \{\{2j + 1, 2j + 2\} : 0 \le j \le 3\}$. A Kirkman frame $(X, \mathcal{G}, \mathcal{A})$ of type 2^4 is constructed with the block set $\mathcal{A}$ as below:

$$\begin{matrix} 358 & 157 & 148 & 136 \\ 467 & 268 & 237 & 245 \end{matrix}$$

Let $\pi = (5\,8)(6\,7)$. Then $(X, \mathcal{G}, \mathcal{A})$ and $(X, \mathcal{G}, \pi(\mathcal{A}))$ both have a holey parallel class $\{3\,5\,8\}$, $\{4\,6\,7\}$ corresponding to hole $\{1\,2\}$. It is readily checked that $|\mathcal{A} \cap \pi(\mathcal{A})| = 2$ and $|\mathcal{A} \cap \mathcal{A}| = 8$. □

Lemma 4.11. *For any* $k \in \{0, 12, 20, 41, 92, 140, 170\}$, *there exists a pair of Kirkman frames with a common holey parallel class of type* 6^6 *having* k *other common blocks.*

Proof. A Kirkman frame $(X, \mathcal{G}, \mathcal{A})$ of type 6^6 is listed as follows:

Point set $X = (((Z_5 \times \{1, 2\}) \cup \infty) \times Z_3) \cup \{a, b, c\}$.

Group set $\mathcal{G}$: $G_i = \{(i, 1, 0), (i, 1, 1), (i, 1, 2), (i, 2, 0), (i, 2, 1), (i, 2, 2)\}$ for $i \in Z_5$, and $G_\infty = \{(\infty, 1), (\infty, 2), (\infty, 0), a, b, c\}$.

Partial parallel classes P_{ij} corresponding to G_i, where $i \in Z_5, j \in Z_3$:

$\{(\infty, j), (i + 3, 1, j), (i + 2, 2, j + 1)\}$,
$\{(\infty, j + 1), (i + 1, 1, j), (i + 3, 1, j + 2)\}$,
$\{(\infty, j + 2), (i + 4, 2, j + 1), (i + 3, 2, j + 2)\}$,
$\{(i + 1, 2, j + 2), (i + 3, 2, j + 1), (i + 4, 2, j + 2)\}$,
$\{(i + 2, 1, j + 1), (i + 3, 1, j + 1), (i + 4, 1, j + 2)\}$,
$\{(i + 3, 2, j), (i + 4, 2, j), (i + 1, 1, j + 2)\}$,
$\{(i + 4, 1, j), (i + 2, 1, j), (i + 1, 2, j)\}$,
$\{a, (i + 1, 1, j + 1), (i + 2, 2, j + 2)\}$,
$\{b, (i + 2, 1, j + 2), (i + 1, 2, j + 1)\}$,
$\{c, (i + 4, 1, j + 1), (i + 2, 2, j)\}$.

Partial parallel classes Q_j corresponding to G_∞, where $i \in Z_5, j \in Z_3$:

$\{(i, 1, j), (i + 4, 1, j + 1), (i + 1, 2, j)\}$,
$\{(i, 1, j + 2), (i + 1, 2, j + 1), (i + 3, 2, j + 2)\}$.

Block set $\mathcal{A} = \bigcup_{j\in Z_3}((\bigcup_{i\in Z_5} P_{ij}) \cup Q_j)$.

Consider the following permutations:

$\pi_0 = ((0, 1, 0)(0, 1, 1))((0, 1, 2)(0, 2, 0))((0, 2, 1)(0, 2, 2))((1, 1, 0)(1, 2, 0))((1, 1, 1)(1, 1, 2))$
$((1, 2, 1)(1, 2, 2))((2, 1, 0)(3, 1, 2))((2, 1, 1)(3, 1, 0))((2, 1, 2)(3, 2, 1))((2, 2, 0)(3, 2, 2))$
$((2, 2, 1)(3, 1, 1))((2, 2, 2)(3, 2, 0))((4, 1, 0)(\infty, 1))((4, 1, 1)(\infty, 2))((4, 1, 2)(\infty, 0))$
$((4, 2, 0)\ a)((4, 2, 1)\ c)((4, 2, 2)\ b)$,
$\pi_1 = ((0, 1, 2)(0, 2, 2))((1, 1, 0)(1, 2, 0))((1, 1, 1)(1, 2, 2))((1, 1, 2)(1, 2, 1))((2, 1, 0)(3, 1, 2))$
$((2, 1, 1)(3, 1, 0))((2, 1, 2)(3, 2, 0))((2, 2, 0)(3, 2, 2))((2, 2, 1)(3, 1, 1))((2, 2, 2)(3, 2, 1))$
$((4, 1, 0)(\infty, 1))((4, 1, 1)(\infty, 2))((4, 1, 2)(\infty, 0))((4, 2, 0)\ b)((4, 2, 1)\ c)((4, 2, 2)\ a)$,
$\pi_2 = ((1, 1, 0)(1, 2, 0))((3, 2, 1)(3, 2, 2))((4, 2, 0)(4, 2, 1))((1, 1, 2)(1, 2, 2))$
$((2, 1, 0)(2, 2, 0))(a, b)((0, 1, 0)(0, 2, 2))((0, 1, 1)(0, 2, 1))((0, 1, 2)(0, 2, 0))$,
$\pi_3 = ((3, 1, 0)(3, 1, 1)(3, 1, 2)(3, 2, 0)(3, 2, 1)(3, 2, 2))$
$((1, 2, 1)(1, 2, 2))((2, 2, 0)(2, 2, 1))((4, 1, 0)(4, 2, 0))$,
$\pi_4 = ((1, 1, 2)(1, 2, 2))((2, 1, 0)(2, 2, 0))(a\, b)$,
$\pi_5 = ((0, 1, 0)(0, 1, 1))$.

It is readily checked that

$|\mathcal{A} \cap \pi_0\mathcal{A}| = 10$ with a common holey parallel class P_{00};
$|\mathcal{A} \cap \pi_1\mathcal{A}| = 22$ with a common holey parallel class P_{00};
$|\mathcal{A} \cap \pi_2\mathcal{A}| = 30$ with a common holey parallel class P_{01};
$|\mathcal{A} \cap \pi_3\mathcal{A}| = 51$ with a common holey parallel class P_{31};
$|\mathcal{A} \cap \pi_4\mathcal{A}| = 102$ with a common holey parallel class P_{01};
$|\mathcal{A} \cap \pi_5\mathcal{A}| = 150$ with a common holey parallel class P_{00};
$|\mathcal{A} \cap \mathcal{A}| = 180$ with a common holey parallel class P_{00}. □

Lemma 4.12. *$I_1[u] \setminus \{t_u - 13, t_u - 7, t_u - 4\} \subseteq J_1[u]$ for $u = 11, 13, 17, 19$.*

Proof. There is a B$((3u-1)/2, 4, 1)$ for $u = 11, 17, 19$ (see [4]). Delete a point of the BIBD to get a 4-GDD $(X, \mathcal{G}, \mathcal{B})$ of group type $3^{(u-1)/2}$ which contains $3(u-1)(u-3)/16$ blocks of size 4. Select a point in the $(u-1)/2$-th group, say f, there are $(u-3)/2$ blocks containing the point f and $(3u-11)(u-3)/16$ blocks not containing the point f in the 4-GDD. Give each point of the GDD weight 2. For each $B \in \mathcal{B}$ containing f, there is a pair of Kirkman frames of type 2^4 with a common holey parallel class which intersect in 0 or 6 other common triples, whose existence is guaranteed by Lemma 4.10. For each block $B \in \mathcal{B}$ not containing f, there is a pair of Kirkman frames of type 2^4 with k_B common triples where $k_B \in \{0, 2, 8\}$, which is guaranteed by Lemma 4.2. Apply Theorem 2.3 to obtain a pair of Kirkman frames of type $6^{(u-1)/2}$ with a common holey parallel class corresponding to the $(u-1)/2$-th group which intersect in $\sum_{1\le i\le (u-3)/2}\alpha_i + \sum_{1\le j\le (3u-11)(u-3)/16}\beta_j$ other triples where $\alpha_i \in \{0, 6\}$ for $1 \le i \le (u-3)/2$ and $\beta_j \in \{0, 2, 8\}$ for $1 \le j \le (3u-11)(u-3)/16$. Use Theorem 2.4 with $a = 3$ to fill the holes of the newly constructed Kirkman frames of type $6^{(u-1)/2}$ with a common holey parallel class corresponding to the $(u-1)/2$-th group. Fill the i-th $(1 \le i \le (u-3)/2)$ group by a pair of KTS(9)s containing the same sub-KTS(3) with b_i common triples where $b_i \in \{1, 2, 3, 4, 6, 12\}$ from [2]; and fill the $(u-1)/2$-th group by a pair of KTS(9)s with a common parallel class that intersect in $b_{(u-1)/2}$ other triples where $b_{(u-1)/2} \in \{0, 3, 9\}$ from Lemma 3.1. By Theorem 2.4 there is a pair of KTS($3u$)s with a common parallel class which intersect in

$$\sum_{1\le i\le (u-3)/2}\alpha_i + \sum_{1\le j\le (3u-11)(u-3)/16}\beta_j + \sum_{1\le i\le (u-1)/2} b_i - \frac{u-3}{2} \tag{5}$$

other common triples where $\alpha_i \in \{0, 6\}$ for $1 \le i \le (u-3)/2$, $\beta_j \in \{0, 2, 8\}$ for $1 \le j \le (3u-11)(u-3)/16$, $b_i \in \{1, 2, 3, 4, 6, 12\}$ for $1 \le i \le (u-3)/2$ and $b_{(u-1)/2} \in \{0, 3, 9\}$. For any $k \in I_1[u] \setminus \{t_u - 13, t_u - 7, t_u - 4\}$ for $u = 11, 17, 19$, it is not difficult to show that k can be written in the form of (5). Hence, $I_1[u] \setminus \{t_u - 13, t_u - 7, t_u - 4\} \subseteq J_1[u]$ for $u = 11, 17, 19$.

For $u = 13$, by Lemma 4.11, there exists a pair of Kirkman frames of type 6^6 with a common holey parallel class corresponding to 6-th group which intersect in k other common triples where $k \in \{0, 12, 20, 41, 92, 140, 170\}$. A similar argument by employing Theorem 2.4 gives a pair of KTS(39)s with a common parallel class which intersect in

$$k + \sum_{1\le i\le 6} b_i - 5$$

other common triples where $k \in \{0, 12, 20, 41, 92, 140, 170\}$, $b_i \in \{1, 2, 3, 4, 6, 12\}$ for $1 \le i \le 5$ and $b_6 \in \{0, 3, 9\}$. It is not difficult to show that $I_1[13] \setminus \{t_{13} - 13, t_{13} - 7, t_{13} - 4\} \subseteq J_1[13]$. □

Lemma 4.13. *Let $u \equiv w \equiv 1 \pmod 2$ and $u \ge 3w$. If $k \in J_1[w]$, then $k + t_u - t_w \in J_1[u]$.*

Proof. From [8], there exists a KTS($3u$) $(X, \mathcal{B})$ containing a sub-KTS($3w$) $(Y, \mathcal{A})$. Replace the sub-KTS($3w$) $(Y, \mathcal{A})$ by a pair of sub-KTS($3w$)s $(Y, \mathcal{A}_1)$ and $(Y, \mathcal{A}_2)$ with a common parallel class $\mathcal{P}$. The parallel class containing $\mathcal{P}$ in the newly KTS($3u$)s is denoted by $\mathcal{P}'$. Suppose $|\mathcal{A}_1 \cap \mathcal{A}_2| = k + w$, the newly pair of KTS($3u$)s with a common parallel class $\mathcal{P}'$ have $u(3u-1)/2 - w(3w-1)/2 + k + w - u (= k + t_u - t_w)$ other common triples. Hence, $k + t_u - t_w \in J_1[u]$. □

Lemma 4.14. *$J_1[u] = I_1[u]$ for any $u \equiv 1, 3 \pmod 8$ and $u \ge 25$; $I_1[u] \setminus \{t_u - 7\} \subseteq J_1[u]$ for $u = 17, 19$.*

Proof. For $u \equiv 1, 3 \pmod 8$ and $u \ge 25$, by Lemma 4.7, $\{0, 1, \ldots, t_u - 20\} \subseteq J_1[u]$. By Lemmas 3.1, 3.2 and 3.11, $J_1[5] = I_1[5] \setminus \{t_5 - 7, t_5 - 6\}$, $t_3 - 6 = 3 \in J_1[3]$ and $t_7 - 7 = 56 \in J_1[7]$. Taking $w = 3, 5, 7$ in Lemma 4.13, we have $k + t_u - t_w \in J_1[u]$ where $k \in J_1[w]$. It is readily checked that $\{t_u - 19, t_u - 18, \ldots, t_u - 6, t_u - 4, t_u\} \subseteq J_1[u]$. Hence, $J_1[u] = I_1[u]$.

For $u = 17, 19$, by Lemma 4.12, we only need to show that $t_u - 13, t_u - 4 \in J_1[u]$. Since $t_5 - 13, t_5 - 4 \in J_1[5]$ from Lemma 3.2, by applying Lemma 4.13 with $w = 5$, we get $k + t_u - t_5 \in J_1[u]$ where $k = t_5 - 13,\ t_5 - 4$. This gives $t_u - 13, t_u - 4 \in J_1[u]$. □

5. Main result

We are in position to present the following main result.

Theorem 5.1. *$J_1[u] = I_1[u]$ for any $u \equiv 1 \pmod 2$, $u \geq 7$ and $u \neq 9, 11, 13, 17, 19$.*

$$J_1[3] = \{0, 3, 9\}; \qquad J_1[5] = I_1[5] \setminus \{23, 24\};$$
$$I_1[u] \setminus \left\{3\binom{u}{2} - 13, 3\binom{u}{2} - 7, 3\binom{u}{2} - 4\right\} \subseteq J_1[u] \quad \textit{for } u = 9, 11, 13;$$
$$I_1[u] \setminus \left\{3\binom{u}{2} - 7\right\} \subseteq J_1[u] \quad \textit{for } u = 17, 19.$$

Proof. For $u \equiv 1 \pmod 2$, $u \geq 7$ and $u \neq 9, 11, 13$, the conclusion follows by Theorem 4.5, Lemmas 3.11 and 4.14. For $u \leq 13$, it follows by Lemmas 3.1, 3.2, 4.9 and 4.12. □

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